Moments and characteristic function of a nonstationary particle distribution after injection

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This paper studies the injection process into a storage ring and presents an analytic model for the nonstationary particle distribution after mismatched or off-axis injection. The effects of nonlinear fields as well as the coupling to the longitudinal motion are described by analytic expressions for the first moments of the particle distribution. The result contains two distinct approximations: first, the Hamiltonian has been replaced by a Hamiltonian averaged over the phase variable; second, the characteristic function of the longitudinal distribution has been confined to the first two cumulants. Functions of moments that remain invariant for this averaged Hamiltonian are constructed.

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I. INTRODUCTION

The moment description for a particle beam is relevant, since the moments correspond to measurable quantities. The behavior of the moments of a particle distribution as it is transported through a Hamiltonian system, e.g., the focusing channel of a storage ring or a linac, has been investigated in the past by several authors. A systematic treatment of the moments and the moment invariants for linear Hamiltonian systems can be found in Ref. [1]. The moment dynamics and moment invariants, based on the Lie-Poisson structure of the Vlasov equation, are discussed in Ref. [2].

In this paper, we deal with moments in the more specific context of the injection process. Suppose the operator in charge of a storage ring has already minimized particle losses at injection by adjusting the transverse tunes (to avoid resonances). In addition, a suitable closed orbit has been chosen, and the operator decides to inject either on or off the closed orbit. Since the injection losses have now been minimized, the effect of nonlinear fields on the beam over a single revolution may be considered small. Nevertheless, the accumulative effect over many turns on the particle distribution may lead to considerable enlargement of beam size and, thus, may degrade the injection efficiency.

The above scenario suggests the use of the averaging method, e.g., the replacement of the actual Hamiltonian $H(I,\phi)$ by a Hamiltonian $\hat{H}(I)$ averaged over the fast evolving canonical variable [3]. The Hamilton-Jacobi perturbation technique has already been used to describe the nonlinear fields in the transverse plane [4,5] in the action variable

$$\hat{H}(I)=\omega_x(I-\mu_x I^2/2)\ , \tag{1}$$

with

$$\mu_x=-\frac{1}{16\nu_x\pi}\oint ds\beta_x^2(s)K_3(s)+\frac{1}{64\nu_x\pi}\oint ds\beta_x^{3/2}(s)K_2(s)\int_s^{s+c}ds'\beta_x^{3/2}(s')K_2(s')\left\{\frac{3\cos[\psi(s')-\psi(s)-\pi\nu_x]}{\sin(\pi\nu_x)}+\frac{\cos[3\psi(s')-3\psi(s)-3\pi\nu_x]}{\sin(3\pi\nu_x)}\right\}\ , \tag{2}$$

where $\beta_x(s)$ denotes the β function [6] and $\psi(s)$ denotes the phase advance. The normalized strength of the sextupoles and octupoles around the ring are described by $K_2(s)$ and $K_3(s)$, and ν_x is the horizontal tune.

A Taylor expansion of the sinusoidal rf wave form around the synchronous phase ϕ_s leads to an averaged Hamiltonian for the longitudinal motion [7]

$$\hat{H}(I)=\omega_s(I-\mu_s I^2/2)\ ,\qquad \mu_s=-\frac{h^2\alpha}{8R\nu_s}\{1+\tfrac{5}{3}\tan^2(\phi_s)\}\ , \tag{3}$$

where h denotes the harmonic number, α is the momentum compaction factor, and $2R\pi$ is the circumference of the ring. To be specific, we consider a high-energy electron storage ring. The results can be adapted easily to proton rings.

We introduce canonical variables (ξ,n) which are related to the measurable transverse coordinates (x_β,p_β) of the betatron motion and the longitudinal coordinates (ε,z) by

Longitudinal		Transverse
$\varepsilon\sqrt{\alpha R/\nu_s}$	$\Longleftarrow\xi\Longrightarrow$	$\frac{x_\beta}{\sqrt{\beta}}$,
$z\sqrt{\nu_s/\alpha R}$	$\Longleftarrow\eta\Longrightarrow$	$\frac{\alpha x_\beta+\beta p_\beta}{\sqrt{\beta}}$,

(4)

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where ε denotes the relative energy deviation and z describes the longitudinal position with respect to the synchronous particle. Their relation to the action-angle variables is given by

$$\eta=\sqrt{2I}\,\cos(\phi),\quad \xi=\sqrt{2I}\,\sin(\phi)\ . \tag{5}$$

A typical initial condition for a Gaussian particle distribution in phase space is illustrated in Fig. 1. The corresponding distribution function in (I,ϕ) at $t=0$ looks like

$$\Psi=\frac{\sqrt{b^2-c^2}}{2\pi}\exp\{-(b+c)[\sqrt{I}\cos(\Omega)-\sqrt{I_0}\cos(\Omega_0)]^2 \\ -(b-c)[\sqrt{I}\sin(\Omega)-\sqrt{I_0}\sin(\Omega_0)]^2\}\ , \tag{6}$$

with

$$\Omega=\phi-\bar{\phi},\quad \Omega_0=\phi_0-\bar{\phi}\ ,$$

$$I_0=\tfrac{1}{2}(\eta_0^2+\xi_0^2),\quad \phi_0=\arctan(\xi_0/\eta_0)\ ,$$

where the center of mass at injections is given by the coordinates η_0, ξ_0. The coefficients b and c describe the injected beam ellipse in the lattice of the storage ring. In the transverse case they are composed of the Twiss parameters associated with the injection point in the storage ring (α,β) and the Twiss parameters that describe the injected beam ellipse α_i, β_i. From Ref. [8] we have

$$b=\frac{1}{2\epsilon_{x0}}\left[\frac{\beta_i}{\beta}+\frac{\beta}{\beta_i}+\frac{\beta}{\beta_i}\left[\alpha_i-\frac{\beta_i}{\beta}\alpha\right]^2\right]\ , \tag{7}$$

and $c,\bar{\phi}$ are given by

$$c\cos(2\bar{\phi})=\frac{1}{2\epsilon_{x0}}\left[\frac{\beta_i}{\beta}-\frac{\beta}{\beta_i}-\frac{\beta}{\beta_i}\left[\alpha_i-\frac{\beta_i}{\beta}\alpha\right]^2\right]\ ,$$

$$c\sin(2\bar{\phi})=\frac{1}{\epsilon_{x0}}\left[\alpha_i-\frac{\beta_i\alpha}{\beta}\right]\ , \tag{8}$$

where ϵ_{x0} denotes the injected emittance. From these equations, we see that

$$b^2-c^2=1/\epsilon_{x0}^2\ .$$

For $c=0$, the initial distribution is described in phase space by circular contours centered around I_0, ϕ_0. A parametrization of the coefficients b, c for the longitudinal plane is given in Ref. [7]. The product of the injected energy spread times bunch length $\sigma_{e0}\sigma_{z0}$ takes in the longitudinal plane the position of the transverse injected emittance.

In the absence of damping and quantum fluctuations, the evolution of the distribution function is governed by

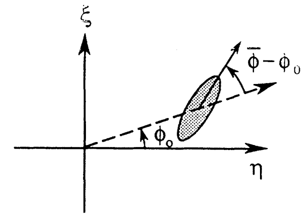

FIG. 1. Injected beam ellipsoid in phase space.

the Liouville equation

$$\frac{\partial\Psi}{\partial t}=\frac{\partial\hat{H}}{\partial I}\frac{\partial\Psi}{\partial\phi}\ . \tag{9}$$

If we replace the phase Ω in Eq. (6) by

$$\Omega=\phi-\int_0^t dt\frac{\partial\hat{H}(I)}{\partial I}-\bar{\phi}=\phi-\omega_0(1-\mu I)t-\bar{\phi}\ , \tag{10}$$

the distribution function in Eq. (6) will be an exact solution of the Liouville equation. The quantity ω_0 denote the revolution frequency times the tune. It is worth noting that the time evolution is not restricted to a Gaussian initial distribution. The only requirement is that the initial particle distribution is well approximated by a positive definite, but not necessarily smooth, function of the phase-space variables.

In the following, we derive and discuss analytic expression for the first and second moments of the distribution function. In Sec. III, we introduce the characteristic function and truncate its expansion in cumulants to second order. Coupling between betatron and synchrotron motion will be discussed in Sec. IV. In order to derive analytic results, we apply the same approximation method that has been used for the characteristic function. Finally, in Sec. V we discuss the injection process in the presence of damping and quantum excitation. We will show that the analytic results for the first and second moments presented in Sec. II are still valid when the effect of damping is included.

II. FIRST AND SECOND MOMENTS

Beam instrumentation provides us with measurements of the first and second moments on successive revolutions after injection. In the Stanford Linear Collider (SLC) damping rings, turn-by-turn data acquisition from beam position monitors and a fast-gated synchrotron light camera are used to analyze the injection process [9,10].

The main results of this article are the analytic expressions in closed form for the first and second moments of the distribution function in Eq. (6) with the time evolution given by Eq. (10). In the Appendix we derive the analytic expression for the first moments.,

$$\langle\eta+i\xi\rangle=\int\int\Psi\sqrt{2I}\,\exp(\phi)dI\,d\phi \\ =\frac{A\sqrt{2I_0(b^2-c^2)}}{\beta_1^2(1-z_1^2)^{3/2}}e^{-I_0[b+c\cos(2\Omega_0)]}e^{i(\omega_0 t+\bar{\phi}+\tilde{\Omega}_0)}[1-z_1e^{-2i\tilde{\Omega}_0}]\exp\{\lambda_1-\lambda_1 z_1\cos(2\tilde{\Omega}_0)\} \tag{11}$$

with

$$\beta_k = b + i(k\omega_0\mu t), \quad z_k = c/\beta_k, \quad \lambda_k = \frac{A^2 I_0}{\beta_k(1-z_k^2)} .$$

Using the definitions of A and $\tilde{\Omega}_0$ in Eq. (A3), it is straightforward to show that Eq. (11) fulfills the initial condition $\langle \eta + i\xi \rangle_{t=0} = \sqrt{2I_0}\exp(i\phi_0)$.

Since $|\beta_1|$ increases with time, the asymptotic value of the first moment tends to zero: $\langle \eta + i\xi \rangle_{t\to\infty} = 0$. In this context one talks about the decoherence of the center-of-mass motion. This effect was observed and analyzed in proton storage rings when the stored beam has been kicked by various angular deflections. Higher-order multipole fields and their effects on the beam were studied by this method at the SPS [11], at the TEVATRON [4], at CESR [12], and recently at the Indiana University Cyclotron Facility [13–15]. With $c=0$ and $\phi_0 = \pi/2$, the distribution function describes the evolution of a beam which has been kicked. Equation (11) reproduces, in this special case, the result presented by Merminga [4] and Meller [16].

The result for the second moments is given by

$$\langle \eta^2 + \xi^2 \rangle = 2\left\{ \frac{b}{b^2 - c^2} + I_0 \right\} \tag{12}$$

and

$$\langle (\eta + i\xi)^2 \rangle = 2\sqrt{b^2 - c^2}\exp\{-I_0[b + c\cos(2\Omega_0)]\} e^{2i(\omega_0 t + \bar{\phi} + \tilde{\Omega}_0)} \times \left\{ \frac{\exp\{\lambda_2 - \lambda_2 z_2 \cos(2\tilde{\Omega}_0)\}}{\beta_2^2(1-z_2^2)^{3/2}} [\lambda_2(1 - z_2 e^{-2i\tilde{\Omega}_0})^2 - z_2 e^{-2i\tilde{\Omega}_0}] \right\} . \tag{13}$$

It is straightforward to extract the single contributions $\langle \eta^2 \rangle$, $\langle \xi^2 \rangle$, and $\langle \eta\xi \rangle$ from Eq. (12) and the complex valued Eq. (13). The proof of these expressions is similar to the proof of Eq. (11). For the second moment, we find the asymptotic relation

$$\langle \xi^2 \rangle_{t\to\infty} = \langle \eta^2 \rangle = \frac{b}{b^2 - c^2} + I_0 , \tag{14}$$

which is a rather important result since it characterizes the amount of beam size enlargement after mismatched and off-axis injection. Clearly, in the operation of a storage ring, one wants to minimize this quantity at injection. I want to emphasize that, up to this point, the only approximation is due to the averaging over the nonlinear fields in the Hamiltonian. We realize that the combination of moments in Eq. (12) is invariant under the nonlinear transformation, since the right-hand side is time independent. Upon closer inspection of this moment invariant, we formulate the following statement.

Let $\Psi(I,\phi)$ be a given function that describes the particle distribution at $t=0$ and suppose the time evolution of the distribution function is governed by the Hamiltonian $\hat{H}(I,t)$, such that for $t>0$ the evolution of the distribution function is given by $\Psi(I, \phi - \int_0^t \partial\hat{H}/\partial I dt)$. Let us now consider an arbitrary function g which depends only on the action variable. The time evolution of the moment of this function is given by

$$\begin{aligned} \langle g(\eta^2 + \xi^2) \rangle_t = \langle g(I) \rangle_t &= \int \int d(I)\Psi \left[I, \phi - \int_0^t \frac{\partial \hat{H}}{\partial I} dt \right] dI\, d\phi \\ &= \int \int g(I)\Psi(I,\phi') dI\, d\phi' \\ &= \langle g(\eta^2 + \xi^2) \rangle_{t=0} , \end{aligned}$$

such that the moment of g remains invariant.

In this context, we give the expression for a quantity which will be used later in the discussion of coupling between longitudinal and transverse motion

$$\begin{aligned} \langle \eta\eta_0 \rangle &= 2\int \int \Psi I \cos(\phi)\cos[\phi - \omega(I)t] dI\, d\phi \\ &= \sqrt{b^2 - c^2}\exp\{-I_0[b + c\cos(2\Omega_0)]\} \\ &\quad \times \mathrm{Re}\left\{ \frac{\exp\{i\omega_0 t + \lambda_1[1 - z_1\cos(2\tilde{\Omega}_0)]\}}{\beta_1^2(1-z_1^2)^{3/2}} \{1 + \lambda_1[1 - 2z_1\cos(2\tilde{\Omega}_0) + z_1^2] + e^{+2i\bar{\phi}}[\lambda_1(e^{i\tilde{\Omega}_0} - z_1 e^{-i\tilde{\Omega}_0})^2 - z_1]\} \right\} . \end{aligned} \tag{15}$$

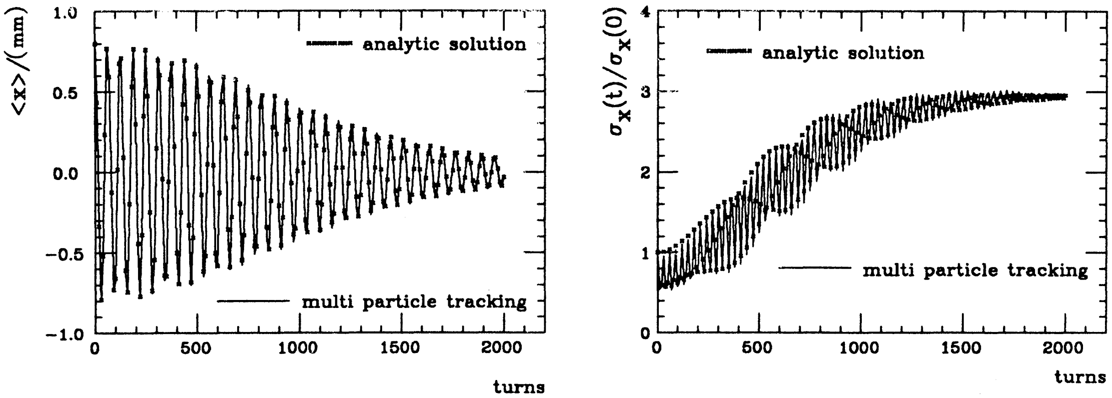

FIG. 2. Center-of-mass and normalized beam size for successive turns after injection.

The proof of this relation follows the same pattern as the proof of Eq. (11).

Figure 2 illustrates the beam size enhancement and the decoherence of the center-of-mass motion due to sextupole fields. One thousand particles have been tracked in the lattice of the SLC damping ring which contains 72 permanent sextupoles for 2000 revolutions. Figure 2 show the center-of-mass motion $\langle x\rangle_t$ in millimeters and the beam size $\sqrt{\langle x^2\rangle_t-\langle x\rangle_t^2}/\sqrt{\langle x^2\rangle_0-\langle x\rangle_0^2}$ normalized by the beam size at injection. The analytic solution is based on Eqs. (11)–(13), and the amplitude dependent tune shift was calculated according to Eq. (2).

III. CHARACTERISTIC FUNCTION AND CUMULANTS OF THE DISTRIBUTION FUNCTION

Besides the distribution function $\Psi(\xi,\eta)$, the characteristic function $\Theta(u_\xi u_\eta)$ describes completely the dynamics of beam distribution in phase space. The distribution function and characteristic function are mutually related by a Fourier transform [17],

$$\Theta(u_\xi,u_\eta)=\int\int\Psi(\xi,\eta,t)e^{iu_\xi\xi+iu_\eta\eta}d\xi\,d\eta$$
$$=\langle e^{iu_\xi\xi+iu_\eta\eta}\rangle\ . \qquad (16)$$

It follows immediately that the moments can be obtained from the characteristic function by differentiation

$$\langle\eta^n\xi^m\rangle=\frac{1}{i^{n+m}}\frac{\partial^{n+m}}{\partial^n u_\eta\partial^m u_\xi}\Theta(u_\xi,u_\eta)\Bigg|_{u_\xi=u_\eta=0}\ .$$

Conversely, given all moments, the characteristic function can be written as a Taylor series

$$\Theta(u_\xi,u_\eta)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(iu_\xi)^m(iu_\eta)^n}{n!m!}\langle\xi^m\eta^n\rangle\ . \qquad (17)$$

For a variety of reasons, it is more convenient to describe the characteristic function by its cumulants

$$\Theta(u_\xi,u_\eta)=\exp\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(iu_\xi)^m(iu_\eta)^n}{m!n!}C_{n+m}(\xi^m\eta^n)\right\}\ , \qquad (18)$$

where the first cumulants are given by

$$C_0=0,$$
$$C_1(x_1)=\langle x_1\rangle,$$
$$C_2(x_1x_2)=\langle x_1x_2\rangle-\langle x_1\rangle\langle x_2\rangle\ .$$

More complicated cumulants can be found in Ref. [18]. In general, it is not possible to solve the integral in Eq. (16). To truncate the series expansion in Eq. (17) at a certain order is dangerous since, in general, the moments obey the relation $\langle x^n\rangle\geq\langle x\rangle^n$. It is, however, convenient to set cumulants higher than a certain order to zero [17]. The choice of the order is considerably simplified by a theorem of Marcinkiewicz [19], which states that the characteristic function cannot be an exponential of a polynomial of degree larger than two. Either we truncate the series in Eq. (18) at $n+m=2$, or we include all terms up to infinity. Marcinkiewicz showed that a function at some order larger than 2 would violate the positive definiteness of the distribution function.

We use the results in Eqs. (11)–(13) to express the cumulants C_1, C_2 and truncate the series in Eq. (18) at $n+m=2$, in order to obtain an approximate expression for the characteristic function

$$\Theta(u_\xi,u_\eta)\approx\exp\{iu_\xi\langle\xi\rangle+iu_\eta\langle\eta\rangle$$
$$-\tfrac{1}{2}[u_\xi^2C_2(\xi^2)+2u_\xi u_\eta C_2(\xi\eta)$$
$$+u_\eta^2C_2(\eta^2)]\}\ . \qquad (19)$$

The introduction and established approximation of the characteristic function will be justified in the following section, where we discuss coupling between longitudinal and transverse motion. We will then use the same method to obtain an approximate solution for the first moments.

IV. COUPLING BETWEEN BETATRON AND SYNCHROTRON MOTION

Up to this point, we considered injection transients in only one degree of freedom. A more realistic description of the injection process into a storage ring includes coupling between the longitudinal and the transverse motion via chromaticity and dispersion. To emphasize the difference with respect to the canonical variables, we will use the rather uncommon notation D_x for the dispersion at the injection point and ν'_x for the chromaticity. The total deviation of an individual particle from the reference orbit of a machine is given by

$$x(t)=D_x\varepsilon+x_\beta=D_x\sqrt{\nu_s/\alpha R}\,\xi_z+\sqrt{\beta_x}\xi_x\ , \qquad (20)$$

where we use the relation Eq. (4) for the relative energy deviation ε and the betatron amplitude x_β. The canonical variables ξ_x, η_x, and ξ_z, η_z are associated with the transverse and longitudinal planes. The first and second moments are then given by

$$\langle x(t)\rangle_{x,z}=D_x\sqrt{\nu_s/\alpha R}\,\langle\xi_z\rangle_{x,z}+\sqrt{\beta_x}\langle\xi_x\rangle_{x,z} \qquad (21)$$

and

$$\langle x^2(t)\rangle_{x,z}=\frac{D_x^2\nu_s}{\alpha R}\langle\xi_z^2\rangle_{x,z}+2D_x\sqrt{\beta_x\nu_s/\alpha R}\,\langle\xi_z\xi_x\rangle_{x,z}+\beta_x\langle\xi_x^2\rangle_{x,z}\ , \qquad (22)$$

where the angular brackets denote integration with respect to the longitudinal and transverse distribution functions as indicated by the subscripts. In the following, we discuss all elements of the right-hand sides of Eqs. (21) and (22).

Integration over the horizontal distribution function in the first terms of Eqs. (21) and (22) leads to a factor of one: $\langle\xi_z\rangle_{x,z}=\langle\xi_z\rangle_z$ and $\langle\xi_z^2\rangle_{x,z}=\langle\xi_z^2\rangle_z$. Integration over the longitudinal distribution function is then given by Eqs. (11)–(13).

The second term in Eq. (22) will only contribute to the beam size if there is a correlation between the transverse and longitudinal distribution functions at $t=0$. A dispersion mismatch would correlate the incoming transverse and longitudinal distribution functions. The Fourier transform of beam size data should then contain a peak at the betatron frequency [20]. This signal was actually observed in the SLC damping rings [10]. A dispersion mismatch at injection is beyond the scope of this work and will be neglected. In the following, we assume $\langle\xi_x\xi_z\rangle=\langle\xi_x\rangle\langle\xi_z\rangle$.

We will focus on the second term in Eq. (21) and the third term in Eq. (22). The transverse tune depends on the amplitude of the transverse action variable and, in addition, on the relative energy deviation of the individual particle. The phase difference after some elapsed time $(t-t_0)$ may be expressed by means of Hamilton's equations

$$\int_{t_0}^{t}\omega_x(I_x,\epsilon)dt=\omega_x(1-\mu_xI_x)(t-t_0)+\nu'_x\frac{c_l}{R}\int_{t_0}^{t}dt\epsilon=\omega_x(1-\mu_xI_x)(t-t_0)-\nu'_x\frac{z-z_0}{R\alpha}\ , \qquad (23)$$

where c_l denotes the speed of light. The center-of-mass motion is given by integrating over the distribution functions

$$\langle n_x+i\xi_x\rangle_{x,z}=\int\int\int\int\Psi_x\Psi_z(\eta_x+i\xi_x)\times dI_xd\phi_xdI_zd\phi_z\ . \qquad (24)$$

We now use the result of Eq. (23) and integrate over the transverse distribution function, which leads to the relation given by Eq. (11) times an exponential containing the chromaticity

$$\langle\eta_x+i\xi_x\rangle_{x,z}=\langle\eta_x+i\xi_x\rangle_x\times\int\int\Psi_ze^{-i\nu'_x(z-z_0)/(R\alpha)}dI_zd\phi_z=\langle\eta_x+i\xi_x\rangle_x\mathcal{E}(\nu'_x/R\alpha)\ .$$

The "envelope" function $\mathcal{E}(u)$ is defined to be the integral over the longitudinal phase space

$$\mathcal{E}(u)=\int\int\Psi_ze^{-iu(z-z_0)}dI_zd\phi_z=\langle e^{-iu(z-z_0)}\rangle_z\ . \qquad (25)$$

Similarly, for the second moments we obtain

$$\langle(\eta_x+i\xi_x)^2\rangle_{x,z}=\langle(\eta_x+i\xi_x)^2\rangle_x\mathcal{E}(2\nu'_x/R\alpha)\ , \qquad (26)$$

$$\langle\eta_x^2+\xi_x^2\rangle_{x,z}=\langle\eta_x^2+\xi_x^2\rangle_x\ , \qquad (27)$$

where the expressions $\langle(\eta_x+i\xi_x)^2\rangle_x$ and $\langle\eta_x^2+\xi_x^2\rangle_x$ are given by Eqs. (12) and (13). To obtain an analytic expression for the envelope function, we proceed in strict analogy to the characteristic function in the previous section. We expand the envelope function by its cumulants C_n and truncate the series at $n=2$,

$$\mathcal{E}(u)\approx\exp\{-iu\langle z\rangle+iu\langle z_0\rangle-\tfrac{1}{2}u^2[C_2(z^2)-2C_2(zz_0)+C_2(z_0^2)]\}\ , \qquad (28)$$

where the brackets denote integration over the longitudinal distribution function. The "mixed" cumulant $C_2(zz_0)=\langle zz_0\rangle_z-\langle z\rangle_z\langle z_0\rangle_z$ is given by Eqs. (11) and (15).

From Eq. (13) we see that the first term on the right-hand side of Eq. (26) goes to zero as t goes to infinity. With Eq. (14) for the asymptotic value of Eq. (27), we obtain the increase of the second moment due to filamentation,

$$\langle x^2(t\to\infty)\rangle_{x,z}=\frac{D^2\nu_s}{\alpha R}\left\{\frac{b_z}{b_z^2-c_z^2}+I_{z0}\right\}+\beta_x\left\{\frac{b_x}{b_x^2-c_x^2}+I_{x0}\right\}\ , \qquad (29)$$

where b_x, c_x, b_z, and c_z denote mismatch parameters in the longitudinal and transverse phase space and I_{x0}, I_{z0} are the action coordinates of the initial beam centroid. This expression equals the asymptotic value of the square of the beam size, since the first moments in Eq. (21) are then zero. The Fourier transform of Eq. (22) contains peaks at the synchrotron sidebands of twice the betatron frequency $2\omega_x\pm2n\omega_s$ which are due to the coupling of the longitudinal to the transverse motion via chromaticity. The asymptotic Eq. (29) for the beam size shows that there is no final beam size enhancement due to this effect.

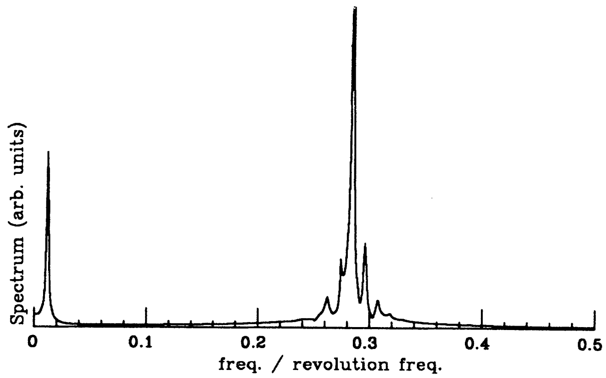

FIG. 3. Power spectrum of the center-of-mass motion after mismatched and off-axis injection.

This result is based on the various assumptions of our analytic model of the injection process.

In Fig. 3, we show the power spectrum of the center-of-mass motion according to the analytic model in Eq. (21). Relevant input parameters for the analytic model are D_x, ν_x', ν_x, ν_s, μ_x, μ_s and ϵ_{x0}, $\sigma_{z0}\sigma_{e0}$, I_{x0}, I_{z0}, b_x, b_z. The tunes were chosen to be $\nu_x=0.285$ and $\nu_s=0.012$. Synchrotron sidebands $\nu_x \pm n\nu_s$ are clearly visible in Fig. 3. A comparison between the analytic model and actual turn-by-turn measurements, which allows, for example, the determination of the amplitude-dependent tune shifts μ_x and μ_s, can be found in Ref. [21].

V. DAMPING AND QUANTUM EXCITATION

Until now we have neglected the effect of synchrotron radiation and the results apply only within a fraction of the damping time after injection. In the presence of damping and quantum excitation, the evolution of the distribution function is governed by the Fokker-Planck equation. The distribution function Eq. (6) with

$$\Omega=\phi-\omega_0 t+f(t)I-\bar{\phi}, \qquad f(t)=\tfrac{1}{2}\omega_0\mu\tau(e^{2t/\tau}-1)\,, \tag{30}$$

is a solution of the Fokker-Planck equation in the limit of no quantum excitation [8]. The coefficients μ, b, c, I_0 in Eqs. (11)–(13) have to be replaced by

$$\omega_0\mu t \Longrightarrow f(t)=\tfrac{1}{2}\omega_0\mu\tau(e^{2t/\tau}-1), \quad b\Longrightarrow\hat{b}(t)=be^{2t/\tau}$$
$$c\Longrightarrow\hat{c}(t)=ce^{2t/\tau}, \quad I_0\Longrightarrow\hat{I}(t)=I_0e^{2t/\tau}\,, \tag{31}$$

where τ denotes the damping time. With this modification, the first and second moment of the distribution function are given by Eqs. (11)–(13). In the presence of quantum excitation, the distribution function of the type of Eq. (6) is no more an exact solution of the Fokker-Planck equation. An approximate solution exists which assumes the injected emittance to be much larger than the equilibrium emittance [8]. In this case, the functions $\hat{c}(t)$, $\hat{b}(t)$, $f(t)$, $\hat{I}(t)$ depend on the damping time and equilibrium emittance.

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APPENDIX: EVALUATION OF THE FIRST MOMENT OF THE DISTRIBUTION FUNCTION

The derivation of Eq. (11) is in two steps. First, we expand part of the distribution functions into a power series of I_0 and c. These terms represent the distance of the center of mass at injection and the deviation from a circular phase-space portrait of the injected ellipse. In this representation, the integration can be performed and leads to a double series involving hypergeometric functions. The second step consists of rejoining all the various contributions to a single analytic expression for the first moment. We start with the distribution function given by Eq. (6),

$$\Psi=\frac{\sqrt{b^2-c^2}}{2\pi}\exp\{-(b+c)[\sqrt{I}\,\cos(\Omega)-\sqrt{I_0}\cos(\Omega_0)]^2 -(b-c)[\sqrt{I}\,\sin(\Omega)-\sqrt{I_0}\,\sin(\Omega_0)]^2\} \tag{A1}$$

with

$$\Omega=\phi-\omega t-\bar{\phi}, \quad \Omega_0=\phi_0-\bar{\phi}\,.$$

We keep in mind that $\omega t=\omega_0 t-f(t)I$ depends on the action variable. The function $f(t)$ equals $\mu\omega_0 t$ in the absence of damping. Otherwise, $f(t)$ is given by Eq. (31). We modify the exponent to obtain

$$\Psi=\frac{\sqrt{b^2-c^2}}{2\pi}\exp\{-Ib-I_0[b+c\cos(2\Omega_0)] -Ic\cos(2\Omega)+2\sqrt{II_0}\,A\cos(\Omega-\tilde{\Omega}_0)\} \tag{A2}$$

with

$$\tan(\tilde{\Omega}_0)=\frac{b-c}{b+c}\tan(\Omega_0), \quad A=\sqrt{b^2+c^2+2cb\cos(2\Omega_0)}\,. \tag{A3}$$

The evaluation of the first moment leads to

$$\langle\eta+i\xi\rangle=\int\int\Psi\sqrt{2I}\,e^{i\phi}d\phi\,dI =\sqrt{b^2-c^2}e^{-I_0[b+c\cos(2\Omega_0)]} \times\int_0^\infty\sqrt{2I}\,R(I)e^{-Ib}dI \tag{A4}$$

with

$$R(I)=\frac{1}{21\pi}\int_0^{2\pi}\exp\{i\phi-ic\cos(2\Omega) +2\sqrt{II_0}\,A\cos(\Omega-\tilde{\Omega}_0)\}d\phi\,.$$

To evaluate this integral, we expand $\exp[2\sqrt{II_0}\,A\cos(\Omega-\tilde{\Omega}_0)]$ in a series and change the integration variable from ϕ to ζ: $\zeta=\Omega-\tilde{\Omega}_0=\phi-\omega t-\bar{\phi}-\tilde{\Omega}_0$, so

$$R(I)=\frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{(2A\sqrt{II_0})^n}{n!}e^{i(\omega t+\bar{\phi}+\tilde{\Omega}_0)}\times\int_0^{2\pi}\exp\{i\zeta-Ic\cos(2\zeta+2\tilde{\Omega}_0)\}\times\cos^n(\zeta)d\zeta\ . \tag{A5}$$

At this point it becomes clear the integral gives a zero contribution if n is even. Hence, we replace n by $2n+1$ and use the identity [22]

$$\cos^{2n+1}(\zeta)=\frac{1}{2^{2n+1}}\sum_{k=0}^{2n+1}\frac{(2n+1)!}{k!(2n+1-k)!}\times\cos(2n\zeta-2k\zeta+\zeta)$$

to obtain

$$R(I)=\frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{k=0}^{2n+1}\frac{(2A\sqrt{II_0})^{2n+1}}{(2n+1-k)!k!}e^{i(\omega t+\bar{\phi}+\tilde{\Omega}_0)}\times\int_0^{2\pi}\exp\{i\zeta-Ic\cos(2\zeta+2\tilde{\Omega}_0)\}\times\cos\{(2n-2k+1)\zeta\}d\zeta\ . \tag{A6}$$

Another change of the integration variable, $2\zeta+2\tilde{\Omega}_0=\theta$, leads to a known integral that may be expressed in terms of Bessel functions

$$R(I)=\tfrac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{2n+1}\frac{(2A\sqrt{II_0})^{2n+1}}{(2n+1-k)!k!}e^{i(\omega t+\bar{\phi}+\tilde{\Omega}_0)}i^{n-k}\times\{iJ_{n-k+1}(icI)e^{-2i(n-k+1)\tilde{\Omega}_0}+J_{n-k}(icI)e^{2i(n-k)\tilde{\Omega}_0}\}\ . \tag{A7}$$

The integral over the action variable in Eq (A4) can be found in [22] and leads to a power series containing hypergeometric functions. To simplify the notation, we define

$$G^{n,k}=\left[\frac{c}{2\beta}\right]^{n-k}\frac{(2n-k+1)!}{\beta^{n+2}\Gamma(n-k+1)}\times F\left[\frac{2n-k+2}{2},\frac{2n-k+3}{2},n-k+1;\left[\frac{c}{\beta}\right]^2\right], \tag{A8}$$

where

$$\beta=b+if(t)\ ,$$

and i denotes the imaginary unit. For the first moments of the distribution, we obtain, using Eq. (A4),

$$\langle\eta+i\xi\rangle=\left[\frac{b^2-c^2}{2}\right]^{1/2}\exp\{-I_0[b+c\cos(2\Omega_0)]+i(\omega_0t+\bar{\phi})\}\times\sum_{n=0}^{\infty}\sum_{k=0}^{2n+1}\frac{(-1)^{n-k+1}[A\sqrt{I_0}]^{2n+1}}{k!(2n-k+1)!}[\exp\{-i(2n-2k+1)\tilde{\Omega}_0\}G^{n,k-1}-\exp\{i(2n-2k+1)\tilde{\Omega}_0\}G^{n,k}]\ . \tag{A9}$$

This expression was already given in [8], but for the analysis of beam position data after injection, the expression it is not very practical. In the process of establishing an approximation to Eq. (A9), either in powers of I_0 or in powers of b, I realized that the contributions of all orders may be summed up in a closed expression. This could be achieved by transforming the hypergeometric functions $F(\alpha,\beta,\gamma,z)$ into a terminating series representation where α and β are negative integers. The key to the treatment of Eq. (A9) is given by the two relations

$$\sum_{n=0}^{\infty}\frac{(A\sqrt{I_0})^{2n}}{\Gamma(n-l+2)(n+l)!}G^{n,l}=\frac{e^{\lambda}}{\beta^2(1-z^2)^{3/2}}\{I_l(\lambda z)+zI_{l-1}(\lambda z)\} \tag{A10}$$

and

$$\sum_{n=0}^{\infty}\frac{(A\sqrt{I_0})^{2n}}{\Gamma(n-l+1)(n+l+1)!}G^{n,l}=\frac{e^{\lambda}}{\beta^2(1-z^2)^{3/2}}\{I_l(\lambda z)+zI_{l+1}(\lambda z)\}\ , \tag{A11}$$

where I_l denotes the modified Bessel function, $l\in Z^+$ and

$$z=\frac{c}{\beta},\quad \lambda=\frac{I_0A^2}{\beta(1-z^2)}\ .$$

Proof of Eq. (A10): From the definition of $G^{n,k}$ in Eq. (A8) and the transformation relation of hypergeometric functions we obtain

$$G^{n,n-l}=\left[\frac{z}{2}\right]^l\frac{(n+l+1)!}{\beta^{n+2}l!(1-z^2)^{n+3/2}}F\left[\frac{l-n}{2},\frac{l-n-1}{2},l+1;z^2\right]\ . \tag{A12}$$

Because of the pole of the Γ function in the denominator of Eq. (A10) for $n-l+2\le 0$, the lower limit of the summation index n is shifted from zero to $n=l-1$. Hence, at least one of the first two coefficients of the hypergeometric function in Eq. (A12) is a negative integer, and the series terminates. Using Eq. (A12), the left-hand side of Eq. (A10) becomes

$$\sum_{n=l-1}^{\infty}\frac{(A\sqrt{I_0})^{2n}}{(n-l+1)!(n+l)!}G^{n,l}=\frac{1}{\beta^2(1-z^2)^{3/2}}\sum_{n=l-1}^{\infty}\lambda^n\frac{(n+l+1)!}{(n+l)!(n-l+1)!l!}\times\left\{\left[\frac{z}{2}\right]^l+\frac{(n-l)(n-l+1)}{(l+1)}\left[\frac{z}{2}\right]^{l+2}+\cdots+\frac{(n-l+1)!l!}{(l+k)!k!\Gamma(n-l-2k+2)}\left[\frac{z}{2}\right]^{l+2k}+\cdots\right\}. \quad \text{(A13)}$$

The next step is to sum over n for fixed index k. For $k=0$ we use the power series expansion of the exponential and obtain

$$\left[\frac{z}{2}\right]^l\frac{1}{l!}\sum_{n=l-1}^{\infty}\frac{\lambda^n(n+l+1)}{(n-l+1)!}=\frac{1}{l!}\left[\frac{z\lambda}{2}\right]^l e^{\lambda}\left[1+2\frac{l}{\lambda}\right].$$

Similarly, we have, for the arbitrary term,

$$\left[\frac{z}{2}\right]^{2k+l}\frac{1}{k!(l+k)!}\sum_{n=l+2k-1}^{\infty}\frac{\lambda^n(n+l+1)}{(n-l-2k+1)!}=\frac{1}{k!(l+k)!}\left[\frac{z\lambda}{2}\right]^{l+2k}e^{\lambda}\left[1+2\frac{l+k}{\lambda}\right]. \quad \text{(A14)}$$

Next, we add up the various contributions over the index k. Inserting the derived contributions of Eq. (A14) into Eq. (A13) we obtain

$$\sum_{n=l-1}^{\infty}\frac{(A\sqrt{I_0})^{2n}}{(n-l+1)!(n+l)!}G^{n,l}=\frac{1}{\beta^2(1-z^2)^{3/2}}\exp\{\lambda\}\sum_{k=0}^{\infty}\left\{\frac{1}{k!(l+k)!}\left[\frac{z\lambda}{2}\right]^{l+2k}+\frac{1}{k!(l+k-1)!}\left[\frac{z\lambda}{2}\right]^{l+2k-1}\right\}.$$

Comparing the right-hand side with the power-series representation of the modified Bessel function, gives the result shown on the left-hand side of Eq. (A10). *End of proof of Eq. (A10).*

Equation (A11) may be shown in a similar way

Let us now go back to the original double series representation of the first moments in Eq. (A9). In order to replace $G^{n,k-1}$ and $G^{n,k}$ by $G^{n,n-l}$, we substitute for k either $k=n-l+1$ or $k=n-l$. Using the transformation relations for hypergeometric functions, we see that $G^{n,n-l}=G^{n,n+l}$ holds and we obtain for the left-hand side of Eq. (A9),

$$\langle\eta+i\xi\rangle=\left[\frac{b^2-c^2}{2}\right]^{1/2}e^{-I_0[b+c\cos(2\Omega_0)]+i(\omega_0 t+\bar{\phi}+\tilde{\Omega}_0)}\times 2\sum_{n=0}^{\infty}(A\sqrt{I_0})^{2n+1}\left\{\frac{G^{n,n}}{(n+1)!n!}+\sum_{l=1}^{\infty}(-1)^l G^{n,n-l}\left[\frac{\exp\{-2il\tilde{\Omega}_0\}}{\Gamma(n-l+2)(n+l)!}+\frac{\exp\{2il\tilde{\Omega}_0\}}{\Gamma(n-l+1)(n+l+1)!}\right]\right\}. \quad \text{(A15)}$$

Substituting Eqs. (A10) and (A11) into Eq. (A15), we obtain

$$\langle\eta+i\xi\rangle=A\sqrt{2I_0(b^2-c^2)}e^{-I_0[b+c\cos(2\Omega_0)]}e^{i(\omega_0 t+\bar{\phi}+\tilde{\Omega}_0)}\times\frac{e^{\lambda}}{\beta^2(1-z^2)^{3/2}}\left\{I_0(z\lambda)+zI_1(z\lambda)+\sum_{l=1}^{\infty}(-1)^l[e^{-2il\tilde{\Omega}_0}\{I_l(z\lambda)+zI_{l-1}(z\lambda)\}+e^{2il\tilde{\Omega}_0}\{I_l(z\lambda)+zI_{l+1}(z\lambda)\}]\right\}.$$

At this point, the summation over the index l may be replaced by the generating function of the Bessel functions to give

$$\langle\eta+i\xi\rangle=A\sqrt{2I_0(b^2-c^2)}e^{-I_0[b+c\cos(2\Omega_0)]}e^{i(\omega_0 t+\bar{\phi}+\tilde{\Omega}_0)}\frac{e^{\lambda}}{\beta^2(1-z^2)^{3/2}}(1-ze^{-2i\tilde{\Omega}_0})\exp\{-\lambda z\cos(2\tilde{\Omega}_0)\},$$

which is the relation we wanted to prove. Higher moments and correlation functions may be treated similarly. Nevertheless, this approach seems to be restricted to a Hamiltonian of the form $H(I)=\omega(I-\mu I^2/2)$ where higher-order contributions of I have been neglected. In order to evaluate moments of distributions, whose evolutions are governed by Hamiltonians of a more general form, it would be of great vlaue to find a more direct and simple approach. Certainly, it is possible to replace the summation over n by modified Bessel functions, in Eq. (A7), but again, the subsequent integration over the action variable is rather troublesome.

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